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Section-and-Etch Study of Hertzian Fracture Mechanics

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A section-and-etch technique is used to study the mechanics of Hertzian cone-crack growth in glass. With a suitable choice of test environment the cone cracks propagate at a convenient rate through successive phases of stability at constant indenter load. Systematic measurements of the etched-crack lengths as a function of indentation time permit a detailed description of the fracture mechanics. Within a certain range of indenter load, the cone crack is observed to grow as a shallow surface ring to a critical depth prior to full development. This directly confirms a salient feature of the energy balance theory of Hertzian fracture outlined in earlier papers. The current status of the long-standing Auerbach law, which relates the critical fracture load linearly to the indenter radius, is discussed in the light of the present evidence.

I. INTRODUCTION

Hertz[1] was the first to give a scientific description of the cone-shaped crack that forms in a brittle solid critically loaded with a spherical indenter. He observed the crack to spread downward into the specimen from just outside the contact circle into a cone of increasing base diameter. Some ten years after Hertz, Auerbach[2] showed that the critical load for cone-crack initiation varies linearly with the radius of the indenting sphere. This relationship, known now as Auerbach's law, is of special interest to fracture theorists because it predicts a "size effect"; the smaller the indenter, the higher is the stress required to initiate the fracture. The theoretical justification of Auerbach's law has consequently been the target of many treatments of Hertzian fracture mechanics.

Recently,[3,4] calculations of the energetics of cone-crack propagation through the inhomogeneous Hertzian stress field have shown Auerbach's law to follow as a direct consequence of the Griffith[5] energy-balance criterion. Since the Griffith concept is essentially an expression of the first law of thermodynamics, these calculations have a sound physical basis. An attractive feature of the energy-balance theory is the appearance of the fracture surface energy in the Auerbach "constant" of proportionality.[3] This has opened up the possibility of using the Hertzian test as a tool for measuring strength variations in brittle materials.[6]

However, certain aspects of the above energy-balance treatment have been questioned by proponents of an alternative "flaw-statistical" explanation of the size effect. This issue, which provides the motivation for the present study, will receive detailed attention in Sec. V. In a previous paper,[4] it was pointed out that the energy-balance theory is characterized by two distinguishing predictions: (i) that Auerbach's law should be independent of size and distribution of initiating flaw, and (ii) that within the limits of validity of the law, the cone crack should grow in a stable manner just prior to its final development. While the first prediction was essentially confirmed by indenting glass surfaces containing flaws introduced in a controlled manner, no distinctive evidence substantiating the second was obtained.

Thus, the present standing of the energy-balance theory of Hertzian fracture may be likened to that of the dislocation theory of plastic flow before techniques were developed for making individual dislocations visible. In this paper, we describe an application of one of the most powerful of the dislocation-revealing methods, namely, the etch method, to the problem of detecting the "embryo" ring cracks. A light etch of specimens sectioned through the indentation diameters delineates clearly the traces of small-scale fractures undetectable by ordinary optical means. Moreover, in analogy with the well-known etch-pit method of determining the mechanics of dislocation motion, straigntforward measurements of cone-crack lengths as a function of indentation load and time are thereby made possible. These observations provide *direct* evidence for the validity of the energy-balance calculations of the mechanics of Hertzian fracture.

II. THEORETICAL BACKGROUND

The curve shown in Fig. 1 represents a statement of the Griffith energy-balance criterion, expressed in terms of Hertzian parameters.[3] A point on the curve indicates the indenter load P for which a conical crack of length c starting from the elastic contact circle is in equilibrium. By increasing the load, the point (P,c) is carried to the right of the curve, whereupon the Griffith condition becomes satisfied and the cone crack is free to extend downward into the specimen until equilibrium is once more attained. It follows that branches c_0 and c_2 represent *unstable* equilibria, while c_1 and c_3 represent *stable* equilibria. Normalization of the load to P_0^* (corresponding to the "hump" in the curve), and the crack length to the contact radius a, makes the plot universal for all indenter sizes.

The equilibrium curve of Fig. 1 strictly applies only to crack growth under environment-free conditions.[6] Auerbach's law then follows if the cone crack initiates from a surface flaw whose characteristic depth

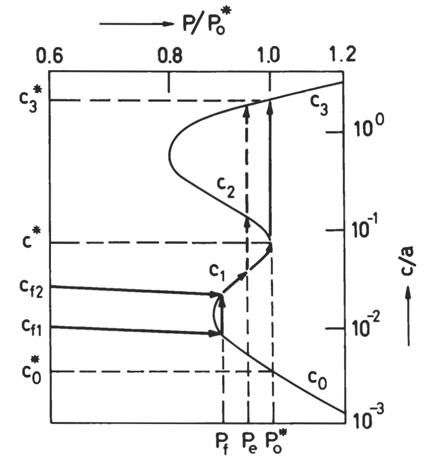

FIG. 1. Equilibrium curve for cone crack propagating downward from contact circle, calculated for $\nu=\frac{1}{3}$. Arrowed solid lines denote crack growth under environment-free conditions (or "instantaneous" loading). Arrowed broken lines denote crack growth within a reactive environment. (Note that because a increases with P, "loading lines" c_f=const have a slight negative slope.)

lies within the range $c_0^* \leq c_f \leq c^*$ (the asterisk denoting values at $P=P_0^*$). The growth sequence from flaw to full cone crack is depicted for two cases ($c_f=c_{f1}, c_{f2}$) by the arrowed solid lines in Fig. 1. The depth of c_f remains unaltered during the initial stages of indenter loading ("loading line"). However, due to the high concentration of tensile stress near the surface of the indented specimen, the flaw is expected to grow into a shallow ring crack around the circle of contact at some point along this loading line.[7] Once the equilibrium curve is intersected at $P=P_f$, downward extension can occur, either unstably from the c_0 to the c_1 branch at constant load or stably along the c_1 branch with subsequent increase in load. The condition that the surface ring propagate into a full cone crack (c_3 branch) then becomes $c_1=c^*$, or [3,4]

$$P_c=P_0^*=K(E)\gamma_0 r \quad \text{(Auerbach's law)},$$

where P_c is the critical load, $K(E)$ is a constant in elastic terms, γ_0 is the fracture energy (the subscript zero denoting environment-free value), and r is the indenter radius. P_0^* is independent of c_{f1}, c_{f2}, hence of all c_f. From this description, it is clear that a detailed observation of crack growth would provide information on only the c_1, and portion of the c_3, branches of the equilibrium curve.

For cone cracks grown in a reactive environment the conditions of equilibrium are somewhat modified. The exact nature of the interaction of the environmental species with the crack tip is not known with certainty, but the over-all effect appears to be one of a rate-controlled subcritical crack growth.[6] The growth sequence for this type of crack behavior is depicted by the arrowed broken lines in Fig. 1 for the simplest possible experimental situation, namely, an initial "instantaneous" loading to $P_e < P_0^*$ followed by subcritical crack growth along P_e = const. The surface ring thus "tunnels" through the hump from the c_1 branch to the critical depth $c=c_2$, whence $P_c=P_e$. In this case, P_c is no longer a simple function of γ_0 and r, depending now on the rate of interaction between environment and crack. Moreover, if $c_f<c_0$ at $P=P_e$, the flaw would need to grow subcritically to the c_0 branch before the crack could tunnel through the hump, in which case P_c would also depend on c_f. However, despite its complicating influence on the mechanics of cone-crack growth, the environmental interaction permits, in principle, a complete description of the equilibrium curve in Fig. 1. Unlike the environment-free case, the ring crack intersects all interposing branches in its growth to full development.

III. EXPERIMENTAL PROCEDURE

Following the procedure outlined in previous papers,[4,6] Hertzian tests were made on glass slabs $2\times2\times\frac{1}{2}$ in., whose surfaces had been abraded in a slurry of No. 400 SiC powder. With a $\frac{1}{2}$-in.-diam steel ball as indenter, the flaws thus introduced fall well within the range of validity of Auerbach's law. The uniformity of flaw distribution within the treated surface also ensures reproducibility in cone-crack geometry and critical load. The specimens were indented on an Instron testing machine. By viewing the contact area through a microscope located beneath the glass slab, successive indents could be accurately centered on a lightly penciled marker line on the test surface. In each test, the indenter was rapidly loaded (rise time $\simeq$1 sec) to $P=P_e$, held constant at this load for a given duration t_D, and then rapidly withdrawn. The loading and unloading could thus be considered "instantaneous" for $t_D \gg 1$ sec.

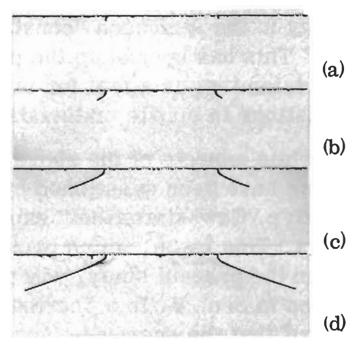

FIG. 2. Etched profile of crack section through diameter of contact. Tests in air, indenter load P_e= 65 kg. Durations of loading t_D as follows: (a) 0.5 sec, (b) 1.4 sec, (c) 1.7 sec, and (d) 100 sec. Note growth of "embryo" crack in (a) and (b) prior to sudden full development. Width of surface ring 860 μm.

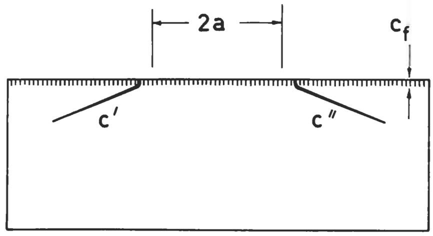

FIG. 3. Geometry of cone-crack profile and abrasion flaws.

Test runs were made in several environments.[6] For each run at a preselected value of P_e, a sequence of varying load durations was imposed. In this way it was hoped to interrupt the cone cracks at various stages of their downward subcritical growth. Working on the assumption that the speed of crack propagation during unstable growth is sufficiently great that the chance of arresting the crack system in a region of instability is negligible,[8] it was plain that measurements of the crack lengths could be used to map out the domain to the left of the equilibrium curve. The specimens were duly sectioned normally to their indented surface, metallographically polished down to the marker lines, and etched in 5% HF for 30 sec. A typical sequence of crack profiles is shown in Fig. 2. Since the presence of a reactive environment during testing inevitably leads to contamination of the fresh fracture surfaces, thereby ensuring that perfect healing of the crack interface cannot occur on indenter withdrawal, we may confidently assert that the etched profile represents an accurate delineation of the entire crack length.

However, one complication in the analysis of the crack profiles demanded particular attention. Cracks approaching instability tended to break through to the next branch from one point just outside the circle of contact, giving rise in some cases to an asymmetric profile. By carefully aligning the indenter ram along the surface normal of the glass specimen, thus avoiding a skew component in the stress field, this effect could be minimized, although never completely eliminated. Small departures from perfect conical symmetry appear to be a consequence of an inherent eccentricity in the initial growth of the surface ring around the circle of contact.[3] An "average" crack length

$$c=\tfrac{1}{2}(c'+c'')$$

was thus derived from the arm lengths c', c'' for each profile (Fig. 3). The results in Sec. IV include only cracks whose "asymmetry factor" $|(c'-c'')/(c'+c'')|$ falls below 5%; the incorporation of all data serves only to increase the scatter.

IV. RESULTS

Figure 4 shows the variation of crack length with indentation time for tests at one load, $P_e=65$ kg, in four environments. This plot reveals two features of consequence: (i) There exist "forbidden" gaps corresponding to sudden crack growth from unstable to stable branches of the equilibrium curve; (ii) the curves for different environments simply translate along the (logarithmic) time axis (within the limits of experimental accuracy), indicating changes only in the *rate* of subcritical growth.[6] The reliability of these curves becomes suspect at the low-scale end of the ordinates; for $t_D \lesssim 1$ sec the response time of the Instron recording equipment precludes an accurate measure of indentation time, while for $c \lesssim c_f (\simeq 10\ \mu\text{m})$, it becomes impossible to distinguish the ring cracks from the background surface damage. We may therefore label with confidence the c_3, c_2, and c_1 branches in Fig. 4, but not the c_0 branch. The test environment, acting as it does to control the velocity of the subcritical crack, becomes an agency through which the time scale of the experiment may be conveniently regulated.

From a systematic tabulation of "allowed" crack lengths over a wide range of indentation loads the equilibrium curve may be inferred. In Fig. 5, all such data are plotted collectively as c/a against P_e (a for each load P_e is readily computed from the Hertz elasticity equations[1,3]). For comparison with the experimental points, the theoretical curve (Fig. 1) is included in Fig. 5, with the adjustment $P_0^*=100$ kg. While there is numerical discrepancy between theory and experiment in Fig. 5, there is essential agreement as to the form of the equilibrium curve. In particular, the experimental data provide striking confirmation of the predicted hump. Because of its central importance in the energy-balance theory, the hump region was given special attention: With silicone oil and toluene as test media, the subcritical cone crack propagated sufficiently slowly that the bounds of the forbidden gaps

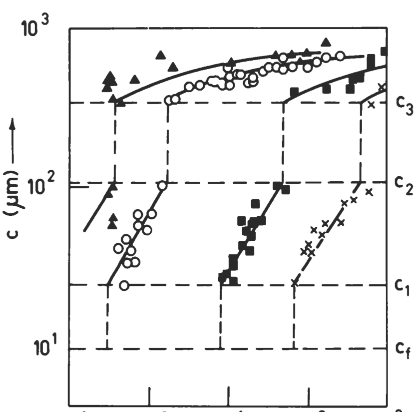

FIG. 4. Cone-crack length c as a function of indentation time t_D for four test environments; water (▲), air (○), toluene (■), silicone oil (×). All data for $P_e=65$ kg. Note discontinuities in growth across "forbidden" gaps, and the similarity in shape of curves for different environments.

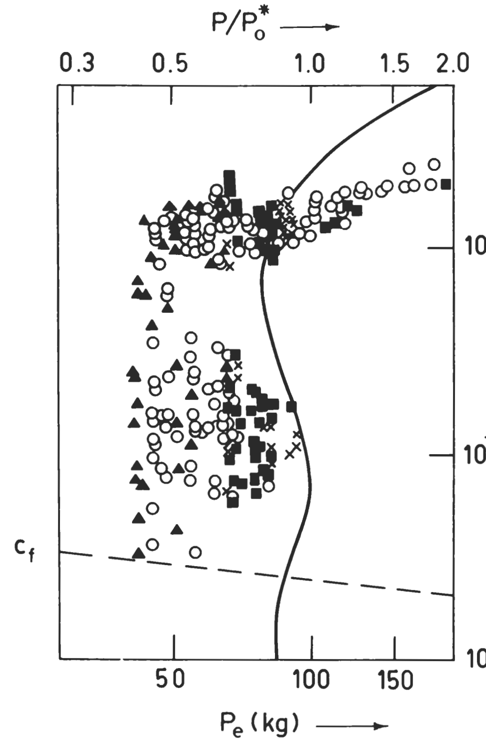

FIG. 5. Cone-crack length c (normalized to contact radius a) as functions of indenter load P_e. Data points for tests in water (▲), air (○), toluene (■), and silicone oil (×). Solid line indicates theoretical curve (Fig. 1). Broken line indicates depth of abrasion flaws.

at loads close to P_0^* could be readily determined within the limits of experimental accuracy.

A quantity which we show later to have special significance is the time taken for the crack to grow subcritically from c_1 to c_2. This "tunnel time" t_{12} is measured directly from curves of the type depicted previously in Fig. 4. A plot of t_{12} as a function of P_e for each of the four test environments used is shown in Fig. 6. It is meaningful to describe a tunnel time only between the limits $P_e \simeq 100$ kg, where c_2 merges with c_1, and $P_e \simeq 50$ kg, where c_2 merges with c_3 (Fig. 5); t_{12} decreases to zero as the upper limit is approached.

V. DISCUSSION

According to the energy-balance theory outlined earlier, the standing of Auerbach's law rests solely on the presence of the hump in the equilibrium curve of Fig. 1. Uncertainty as to the exact shape of the hump affects only the numerical value of the Auerbach constant of proportionality, and not the form of the relationship between P_c and r. However, as pointed out in Sec. I, the energy-balance interpretation has been questioned by other workers.[9–11] Arguments essentially challenging the very existence of the hump region have been formulated on both experimental and theoretical grounds.

From the experimental viewpoint, the present section-and-etch observations appear to provide irrefutable evidence for this key feature of the energy-balance curve. And yet other careful searches using equally sensitive flaw-detection techniques (e.g., ion-exchange treatment,[9] scanning electron microscopy,[11] and optical microscopy[12]) have failed to reveal the slightest trace of the shallow surface ring cracks. Most of these earlier attempts were, however, made in a laboratory environment, and on glass surfaces in their as-received state, conditions which the present results indicate to be highly unfavorable for detecting the embryo growth stages. Figure 6 shows that for air-formed cone cracks, the time for the subcritical ring to tunnel through the hump at constant load is typically of the order of seconds. Now with our abraded glass surfaces we ensure a uniform distribution of surface flaws well within the range $c_0^* \le c_f \le c^*$ (Fig. 1). From such flaws we would expect the growth of a surface ring into the hump region almost immediately upon loading the indenter beyond $P_e \simeq 50$ kg (lower limit to load in hump, Fig. 5). Even for the special case $c_f = c_{f1}$ at $P < P_e$ (Fig. 1), any delay time t_{f0} corresponding to preliminary growth from c_f to c_0 would be expected to be reasonably short, i.e., $t_{f0} < t_{12}$ (Fig. 4). On the other hand, with as-received surfaces the flaws are inevitably distributed widely in both location and size, cone cracks commonly nucleating from relatively small flaws $c_f \ll c_0^*$. In such cases we might encounter extended delay times $t_{f0} \gg t_{12}$, with large scatter for loads $P_e \le P_0^*$. Table I, which lists the times to produce full cone fractures in the same glass slabs as above, but with their surfaces unabraded, substantiates this point. Thus, whereas for abraded surfaces we have accurate foreknowledge in our search for the shallow surface rings, the same is not true for unabraded surfaces. With the latter the erratic fracture behavior militates strongly against a chance arrest of the surface ring during its momentary interlude within the hump just prior to full development. This argument extends to experimental arrangements

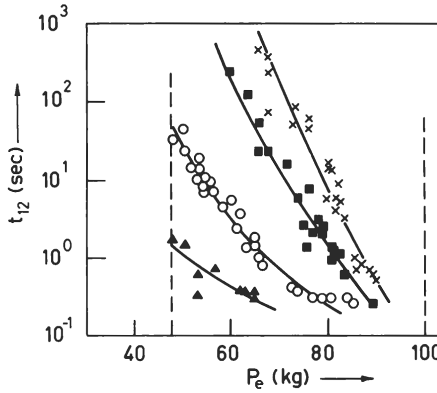

FIG. 6. Time for subcritical crack to tunnel through hump from branch 1 to 2. As $P_e \to 100$ kg, then $t_{12} \to 0$.

TABLE I. Tests on as-received glass surfaces in air using $\frac{1}{2}$-in.-diam steel ball. Times to fracture after loading "instantaneously" to P_e (three values); 20 tests per load.

P_e (kg)	Time to fracture (sec)[a] min	median	max	t_{12} (sec)[b]
100	1.0	12	243	0
90	1.0	36	566	0.1
80	1.4	105	1705	0.3

[a]Neglecting the time spent by the cone crack within the regions of instability, the time to fracture is simply $t_{f0}+t_{12}$.
[b]From Fig. 6, curve for air.

other than the instantaneous loading schedule described in detail here, e.g., monotonic loading of the indenter with time to the point of visible fracture.[6]

As to the theoretical situation there are criticisms which appear to be well founded. For instance, it has been pointed out[11] that although the tensile stress is a maximum at the contact circle, the stress distribution immediately below the surface is apparently such that a crack releases strain energy more rapidly at some more remote starting point.[13] Also, it is claimed that the value for Poisson's ratio used in the present calculations, namely, $\nu=\frac{1}{3}$, is probably too high for glass. On repeating the energy-balance calculations for a wider surface crack and lower ν, the equilibrium curve undergoes considerable changes, most significant of which is a flattening out of the hump. However, because of several simplifying assumptions,[3] e.g., approximation of plane-crack geometry, zero frictional traction at indenter-specimen contact, evaluation of crack-path stresses from stress trajectory patterns, etc., too much reliance should not be placed on the exactness of the calculations. Indeed, the experimental evidence in Fig. 5 indicates a distinct tendency for the theory to underestimate the sharpness of the extrema in the equilibrium curve. There is therefore strong justification for the original energy-balance prediction of the characteristic hump, despite acknowledged shortcomings in the theoretical treatment.

Finally, a contentious point which has been raised in the recent literature, namely, the status of Auerbach's law in Hertzian fracture theory, warrants brief attention here. Hamilton and Rawson,[10] in testing several as-received and etched-glass surfaces, conclude that the linear relationship between P_c and r does not universally apply within the "Auerbach range".[14] They accordingly suggest that the designation "Auerbach's law" should be discontinued. On the other hand, our tests on controlled surfaces strongly substantiate the energy-balance claim that Auerbach's law lies deeply rooted in the principle of energy conservation, and that the so-called Auerbach constant depends only on strength-determining material constants, including the fracture energy. There is therefore good reason for wishing to retain Auerbach's law as a theoretical base for the Hertzian fracture test. Nevertheless, it should be reiterated here that the energy-balance concept does *not* imply universality in the relationship between Hertzian parameters. Unless certain conditions apply, e.g., specimen homogeneous and free from residual stress (cf. severely abraded surfaces,[4] surface-toughened plate glass), environment nonreactive,[6] etc., there may be deviations from Auerbach's law. In particular, for tests on surfaces containing flaws sparsely distributed and outside the special size range $c_0^* \le c_f \le c^*$ (e.g., carefully handled as-received glass surfaces, highly polished or etched surfaces), a nonlinear relationship between P_c and r would be predicted,[3,4] and a flaw-statistical analysis would be appropriate.[15] It is a feature of the energy balance concept that its generality extends beyond the realm of Auerbach's law: It is, in principle, capable of specifying the limits of validity of this law, and of accommodating the factors leading to its breakdown within the theoretical framework.

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[8]Because of the limited regions of instability the propagating cone crack never approached the terminal velocities characteristic of fast-running brittle fractures. In fact, for loads P_e small compared with P_0^*, the unstable growth from c_2 could be easily followed by eye, taking sometimes up to 1 sec to extend to c_3. Nevertheless, the crack speed during instability always greatly exceeded that during subcritical growth, so our assumption may be considered good in an experiment in which only a limited number of data points is recorded. (Note, for example, absence of data points in unstable regions in Fig. 4, later.)

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[11]J. Heavens and T. R. Wilshaw (private communications) have extended the original energy-balance calculations in the manner described in the text. Heavens has also searched for the shallow ring cracks using scanning electron microscopy and other techniques. We are grateful to both of the above for pertinent criticisms of some of the

theoretical points and for making available their results prior to publication.

[12]Earlier attempts made within these laboratories to detect the surface rings through strategically placed microscopes, while the specimen was under load, met with no more success than the attempts by the other workers, no doubt for the same reasons.

[13]In a previous paper (Ref. 4), the relative location of contact circle and surface crack was made visible by lightly smearing the indenter with grease. Heavens has since pointed out that the grease smears more than one might suspect, and that the cone crack consequently starts not at the circle of contact but well outside it, even on abraded surfaces.

[14]The term "Auerbach range" appears to have caused some confusion. Hamilton and Rawson, in their earlier paper (Ref. 9), specify this range as that portion of the $\log_{10}P_c$ vs $\log_{10}r$ curve below the point of intersection of two representative straight lines empirically fitted to the data. Others interpret the Auerbach range to simply indicate that portion of the curve where $P_c \propto r$, i.e., where Auerbach's law is valid. In view of the present evidence this term, if it must be used at all, would perhaps be better restricted to the second convention.

[15]J. D. Poloniecki and T. R. Wilshaw (unpublished).